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1 Joint pdf for PMA(1)

The PMA(1) process is non-Markovian but a one-dependent strictly stationary stochastic process (Watson, 1954). It is straightforward to show that $\{Y_t\} \overset{i.i.d.}{\sim} \mathcal{P}(\sigma,\alpha)$ marginally. The stationary marginal probability density function (pdf) $f_{Y_t}(y)$ follows from the survival function $\overline{F}_{Y_t}(y) = \mathbb{P}(Y_t \geq y) = 1/(1+(y/\sigma)^\alpha)$, and is given by

$$f_{Y_t}(y) = \frac{dF_{Y_t}(y)}{dy} = \frac{\alpha}{\sigma}\big(\frac{y}{\sigma}\big)^{(\alpha-1)}\big(1+\big(\frac{y}{\sigma}\big)^{\alpha}\big)^{-2}. \tag{1}$$

From (1) the mean and variance of $\{Y_t, t\in\mathbb{Z}\}$ are given by, respectively,

$$\mathbb{E}(Y_t) = \int_0^\infty y f_{Y_t}(y)\mathrm{d}y = \sigma\Big(\frac{\pi}{\alpha}\Big)\csc\Big(\frac{\pi}{\alpha}\Big), \tag{2}$$

$$\mathrm{Var}(Y_t) = 2\sigma^2\Big(\frac{\pi}{\alpha}\Big)\csc\Big(\frac{2\pi}{\alpha}\Big) - \mathbb{E}^2(Y_t), \tag{3}$$

where we used the standard integral formula

$$\int_0^\infty \frac{x^a}{(r+sx^b)^{c+1}}\mathrm{d}x = \frac{1}{br^{c+1}}\Big(\frac{r}{s}\Big)^{a/\nu}\frac{\Gamma\big(\frac{a}{b}\big)\Gamma\big(1+c-\frac{a}{b}\big)}{\Gamma(1+c)},\ \Big(0<\frac{a}{b}<c+1, r\neq 0, s\neq 0\Big), \tag{4}$$

with $\Gamma(u)\Gamma(1-u) = \pi\csc(\pi u)$, the cosecant identity $(u\neq 1)$, $\Gamma(2+u) = (1+u)\Gamma(1+u)$. Note that $\csc(z) = 1/\sin(z)$.

A closed form representation of the joint pdf can be obtained for the PMA(1) process by first considering the joint survival function $\overline{F}_{Y_t,Y_{t-1}}(y_1,y_2)$ of (Y_t, Y_{t-1}). That is

$$\begin{aligned}\overline{F}_{Y_t,Y_{t-1}}(y_1,y_2) &= \mathbb{P}(Y_t\geq y_1, Y_{t-1}\geq y_2) = \mathbb{P}(p^{-1/\alpha}\varepsilon_{t-1}\geq y_1, p^{-1/\alpha}\varepsilon_{t-2}\geq y_2)p^2\\ &+\mathbb{P}\big(p^{-1/\alpha}\varepsilon_{t-1}\geq y_1, \min(p^{-1/\alpha}\varepsilon_{t-2},\varepsilon_{t-1})\geq y_2\big)p(1-p)\\ &+\mathbb{P}\big(\min(p^{-1/\alpha}\varepsilon_{t-2},\varepsilon_{t-1})\geq y_1, p^{-1/\alpha}\varepsilon_{t-1}\geq y_2\big)p(1-p)\\ &+\mathbb{P}\big(\min(p^{-1/\alpha}\varepsilon_{t-1},\varepsilon_{t})\geq y_1, \min(p^{-1/\alpha}\varepsilon_{t-2},\varepsilon_{t-1})\geq y_2\big)(1-p)^2.\end{aligned} \tag{5}$$

By the independence of the ε_t's, and letting $z_i = (y_i/\sigma)^\alpha$, $i\in\{1,2\}$, (5) can be written as

$$\overline{F}_{Y_t,Y_{t-1}}(y_1,y_2) = \begin{cases}\dfrac{1}{1+pz_1}\Big(\dfrac{p^2}{1+pz_2}+\dfrac{p(1-p)}{(1+pz_2)(1+z_2)}+\dfrac{p(1-p)}{1+pz_2}+\dfrac{(1-p)^2}{(1+pz_2)(1+z_2)}\Big), & \text{if } 0\leq y_1 < p^{(1/\alpha)}y_2,\\ \dfrac{1}{1+pz_1}\Big(\dfrac{p^2}{1+pz_2}+\dfrac{p(1-p)}{(1+pz_2)(1+z_2)}+\dfrac{p(1-p)}{1+z_1}+\dfrac{(1-p)^2}{(1+z_1)(1+z_2)}\Big), & \text{if } y_1\geq p^{(1/\alpha)}y_2\geq 0.\end{cases} \tag{6}$$

Since $\overline{F}_{Y_t,Y_{t-1}}(y_1,y_2) = 1-F_{Y_t}(y_1)-F_{Y_{t-1}}(y_2)+F_{Y_t,Y_{t-1}}(y_1,y_2)$, it follows that the joint pdf

$f_{Y_t,Y_{t-1}}(y_1,y_2) = \partial^2 F_{Y_t,Y_{t-1}}(y_1,y_2)/\partial y_1 \partial y_2$ is given by

$$f_{Y_t,Y_{t-1}}(y_1,y_2) = \begin{cases} \dfrac{p}{(\sigma/\alpha)^2} \dfrac{z_1^{(1-1/\alpha)}}{(1+pz_1)^2} \dfrac{z_2^{(1-1/\alpha)}}{(1+z_2)^2}, & \text{if } 0 \le y_1 < p^{(1/\alpha)} y_2, \\ \dfrac{1}{(\sigma/\alpha)^2} z_1^{(1-1/\alpha)} \dfrac{z_2^{(1-1/\alpha)}}{(1+z_2)^2} \Big\{ \dfrac{p^3}{(1+pz_1)^2} + \dfrac{(1-p)}{(1+z_1)^2} \Big\}, & \text{if } y_1 \ge p^{(1/\alpha)} y_2 \ge 0. \end{cases} \tag{7}$$

The conditional pdf $f_{Y_t|Y_{t-1}}(y_1|y_2) = f_{Y_t,Y_{t-1}}(y_1,y_2)/f_{Y_{t-1}}(y_2)$ follows from (1) and (7) directly.

2 Autocorrelation for PMA(1)

It is easy to verify that the overall pattern of the ACF of the PMA(1) cuts off after lag one. This property is similar to the cut-off property of the linear MA(1) process with i.i.d. innovations following a symmetric distribution. Using (7) with $\sigma = 1$, $\mathbb{E}[Y_t Y_{t-1}]$ can be compactly written as

$$\mathbb{E}[Y_t Y_{t-1}] = \alpha^2 \Big\{ pA(p,\alpha) + (1-p)B(p,\alpha) + p^3 C(p,\alpha) \Big\}, \tag{8}$$

where

$$A(p,\alpha) = \int_0^\infty \Big(\int_0^{p^{(1/\alpha)} y_2} y_1 \frac{y_1^{1-1/\alpha}}{\big(1+py_1^\alpha\big)^2} \mathrm{d}y_1 \Big) y_2 \frac{y_2^{1-1/\alpha}}{\big(1+y_2^\alpha\big)^2} \mathrm{d}y_2,$$

$$B(p,\alpha) = \int_0^\infty \Big(\int_{p^{(1/\alpha)} y_2}^\infty y_1 \frac{y_1^{1-1/\alpha}}{\big(1+y_1^\alpha\big)^2} \mathrm{d}y_1 \Big) y_2 \frac{y_2^{1-1/\alpha}}{\big(1+y_2^\alpha\big)^2} \mathrm{d}y_2,$$

$$C(p,\alpha) = \int_0^\infty \Big(\int_{p^{(1/\alpha)} y_2}^\infty y_1 \frac{y_1^{1-1/\alpha}}{\big(1+py_1^\alpha\big)^2} \mathrm{d}y_1 \Big) y_2 \frac{y_2^{1-1/\alpha}}{\big(1+y_2^\alpha\big)^2} \mathrm{d}y_2.$$

Explicit representations for each of these terms can be obtained with the help of Mathematica. This approach gives rise to lengthy and complicated expressions involving the Gauss hypergeometric function ${}_2F_1[a,b;c;x]$.

Now using (2) and (3) with $\sigma = 1$, the Pearson ACF at lag 1 of the PMA(1) process is given by

$$\rho(1) = \frac{\mathrm{Cov}(Y_t, Y_{t-1})}{\mathrm{Var}(Y_t)} = \frac{\text{Equation (8)} - \{(\pi/\theta)\csc(\pi/\alpha)\}^2}{(\pi/\alpha)\big(2\csc(2\pi/\alpha) - (\pi/\alpha)\csc^2(\pi/\alpha)\big)}. \tag{9}$$

For $1 < \alpha < 2$ the ACF is not defined.

3 ACVF and ACF for PMA(1): Gini-based results

Assuming that $\{Y_t, t \in \mathbb{Z}\}$ has finite second-order moments, it is not difficult to obtain a closed form representation for the Gini-based (G) autocovariance of lag zero, i.e. $\gamma^{(\mathrm{G})}(0) = \mathrm{Cov}(Y_t, F(Y_t))$. In fact, because the ε_t's are i.i.d., we have

$$\mathbb{E}[Y_t F(Y_t)] = \int_0^\infty \epsilon \Big(\frac{\epsilon^\alpha}{1+\epsilon^\alpha} \Big) \Big(\frac{\alpha \epsilon^{\alpha-1}}{(1+\epsilon^\alpha)^2} \Big) \mathrm{d}\epsilon = \frac{1+\alpha}{2\alpha} \Big(\frac{\pi}{\alpha} \Big) \csc\Big(\frac{\pi}{\alpha} \Big), \tag{10}$$

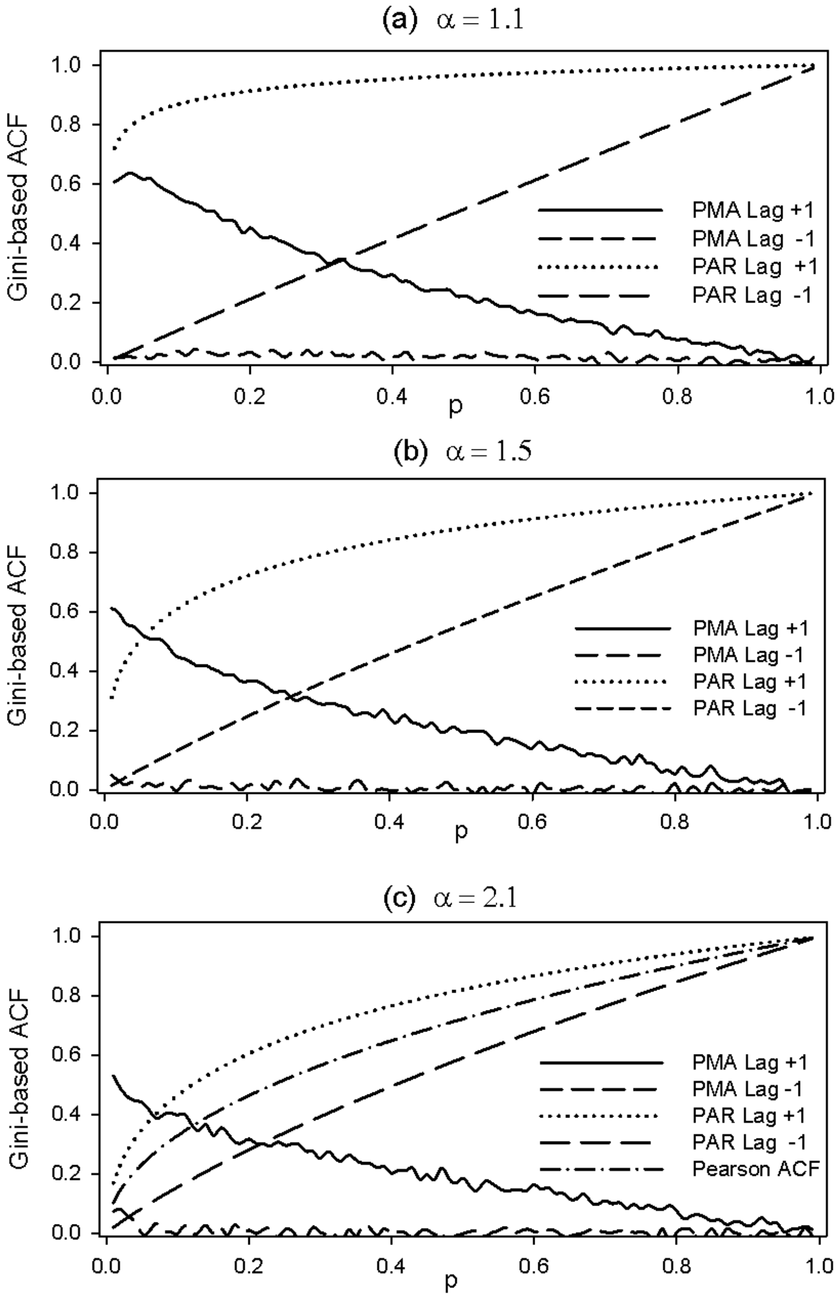

Figure 1: Gini-based ACFs for the PMA(1) and for the PAR(1) process. Figure 1(c) includes the Pearson ACF (dashed-dotted line) for the PAR(1) process with $\alpha = 2.1$.

where in the last step we used (4). Then substituting (2) and (10) into $\gamma^{(G)}(\ell) = 4\big\{\mathbb{E}[Y_t F(Y_{t-\ell})] - \frac{1}{2}\mathbb{E}[Y_t]\big\}$ gives

$$\gamma^{(G)}(0) = \frac{2}{\alpha}\Big(\frac{\pi}{\alpha}\Big)\csc\Big(\frac{\pi}{\alpha}\Big). \tag{11}$$

It is a more challenging task to obtain explicit expressions for $\gamma^{(G)}(\ell)$ for $\ell = \pm 1$. This requires the evaluation of $\mathbb{E}[Y_t F(Y_{t-1})]$ and $\mathbb{E}[Y_t F(Y_{t+1})]$. In practice, however, $\gamma^{(G)}(\ell)$ and $\rho^{(G)}(\ell) = \gamma^{(G_1)}(\ell)/\gamma^{(G)}(0)$, can best be approximated by Monte Carlo simulation.

As an example, consider a type-III PMA(1) process with parameters p and α. We estimated $\rho^{(G)}(\ell)$ as a function of $0 < p < 1$ for $\alpha = 1.1, 1.5$, and 2.1. In each case the number of replications was set at 1,000 and the series length at $n = 10,000$. As a natural estimator of $\gamma^{(G)}(\ell)$, we use

$$\widehat{\gamma}^{(G)}(\ell) = \frac{1}{n-\ell-1}\sum_{t=1}^{n-\ell}(Y_{t+\ell} - \overline{Y}_{((\ell+1):n)})(R(Y_t) - \overline{R}(Y_{(1:(n-\ell))}), \tag{12}$$

where $\overline{Y}_{((\ell+1):n)} = \sum_{t=\ell+1}^{n} Y_t/(n-\ell)$ (sample mean of the last $n-\ell$ observations), $R(Y_t)$ denotes the rank of Y_t (divided by the sample size), and $\overline{R}(Y_{(i:j)}) = \sum_{t=i}^{j} R(Y_t)/(j-i+1)$. Figure 1(a)–(c) shows $\widehat{\rho}^{(G)}(\ell)$ for lags $\ell = \pm 1$.

For comparison purposes, we also plot the lag-one Gini-ACF for a PAR(1) process. Carcea and Serfling (2015) showed that in this case

$$\rho^{(G)}(-1) = \frac{\alpha p(1-p^{1/\alpha})}{1-p}, \quad \rho^{(G)}(1) = \frac{\alpha p(p^{-1/\alpha}-1)}{1-p} = p^{-1/\alpha}\rho^{(G_1)}(-1). \tag{13}$$

Explicit expressions of $\rho^{(G)}(\ell)$ are not available for $\ell \neq \pm 1$.

We see that the overall pattern of the PMA(1) autocorrelations is very different from that of the PAR(1) autocorrelations. For $\alpha = 1.1$ the interquartile range of PMA(1) sample ACFs at lag +1 is equal to [0.09, 0.39] while at lag −1 the interquartile range is much shorter, i.e., [0.01, 0.02]. The interquartile range tends to become smaller as α increases to 1.5 and 2.1. On the other hand, the ACFs for the PAR(1) model have a wide range of values which are quite different from the ones obtained for the PMA(1) model.

We also observe that sets of autocorrelations for the PAR(1) model change considerably with values of p but less with α. Figure 1(c) contains a plot of the lag-one Pearson ACF for the PAR(1) process with $\alpha = 2.1$ (dashed-dotted line). We see that its curve lies between the curves of $\rho^{(G)}(-1)$ and $\rho^{(G)}(+1)$. This indicates that the theoretical Pearson ACF only detects information about the symmetric structure of a time series process in the finite variance case. On the other hand, the Gini autocorrelations allow the data generating process to be asymmetric.

4 Block-length b and size of test statistic $T^{(G)}(m)$

A small simulation study illustrates the influence of the block-length b on the empirical size of the $T^{(G)}(m)$ test statistic, In particular, Table 1 below reports results for $n = 200, 400$, with $\alpha = 1.5$. It can be seen that for both sample sizes the empirical sizes are reasonably stable and relatively close to the 5% nominal significance level. It is interesting to compare these results with the size results reported in Table 1 (column 12) of the main manuscript using the moving block bootstrap procedure of HHL with cross-validation (CV). Clearly, the difference between both bootstrap approaches (fixed block-length versus CV-based block-length) is minimal.

Table 1: Empirical size (in %) of the $T^{(G)}(m)$ test statistic at the 5% nominal significance level and computed from 1,000 independent realizations of a linear MA(1) model of the form $Y_t = Z_t + 0.5Z_{t-1}$, and four fixed block-lengths b.

	$n = 200$				$n = 400$			
m	$b = 10$	$b = 15$	$b = 20$	$b = 25$	$b = 10$	$b = 15$	$b = 20$	$b = 25$
5	4.7	5.3	4.6	5.6	4.3	4.9	3.7	4.3
10	5.6	5.1	5.3	5.1	4.7	5.8	4.8	6.2
20	4.3	5.8	5.6	5.4	4.3	4.3	4.7	4.3

References

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